Problem 2.6

Find the electric field a distance z above the center of a flat circular disk of radius R (Fig. 2.10) that carries a uniform surface charge σ. What does your formula give in the limit $R \to \infty$? Also check the case $z \gg R$.

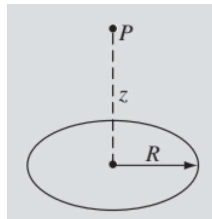
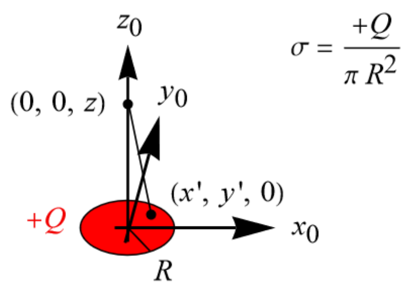


Fig. 2.10

Solution

Start by drawing a schematic for some point on the circular disk.



The formula for the electric field from a continuous distribution of charge on a surface is

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\mathbf{r}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\, da' = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^2}\left(\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\right) da'$$

$$= \frac{1}{4\pi\epsilon_0}\int \frac{\sigma(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}(\mathbf{r}-\mathbf{r}')\, da',$$

where the integral is taken over the surface where the charge exists. Note that $\mathbf{r}$ is the position vector to where we want to know the electric field, $\mathbf{r}'$ is the position vector to the point we chose on the surface, and $\mathscr{r} = |\mathbf{r}-\mathbf{r}'|$ is the distance from the point we chose on the surface to where we want to know the electric field.

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\iint\limits_{x_0^2+y_0^2\le R^2} \frac{\sigma}{\left[\sqrt{(0-x')^2+(0-y')^2+(z-0)^2}\right]^3}(\langle 0,0,z\rangle - \langle x',y',0\rangle)\, dA'$$

The disk is circular, so the appropriate parameterization is done with polar coordinates.

$$\mathbf{r}' = r'\langle \cos\theta', \sin\theta', 0\rangle, \quad 0 \le r' \le R,\ 0 \le \theta' \le 2\pi$$

Consequently, the electric field at $\mathbf{r} = \langle 0,0,z\rangle$ is

$$\mathbf{E} = \frac{\sigma}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^R \frac{1}{\left[\sqrt{(0-r'\cos\theta')^2+(0-r'\sin\theta')^2+(z-0)^2}\right]^3}(\langle 0,0,z\rangle - r'\langle\cos\theta',\sin\theta',0\rangle)(r'\,dr'\,d\theta').$$

Simplify the integrand and then integrate the components.

$$\begin{aligned}
\mathbf{E} &= \frac{\sigma}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^R \frac{1}{(r'^2+z^2)^{3/2}}\langle -r'\cos\theta', -r'\sin\theta', z\rangle (r'\,dr'\,d\theta') \\
&= \frac{\sigma}{4\pi\epsilon_0}\left\langle -\int_0^{2\pi}\int_0^R \frac{r'^2\cos\theta'}{(r'^2+z^2)^{3/2}}\,dr'\,d\theta', -\int_0^{2\pi}\int_0^R \frac{r'^2\sin\theta'}{(r'^2+z^2)^{3/2}}\,dr'\,d\theta', \int_0^{2\pi}\int_0^R \frac{r'z}{(r'^2+z^2)^{3/2}}\,dr'\,d\theta'\right\rangle \\
&= \frac{\sigma}{4\pi\epsilon_0}\left\langle -\left[\int_0^R \frac{r'^2}{(r'^2+z^2)^{3/2}}\,dr'\right]\left(\int_0^{2\pi}\cos\theta'\,d\theta'\right), -\left[\int_0^R \frac{r'^2}{(r'^2+z^2)^{3/2}}\,dr'\right]\left(\int_0^{2\pi}\sin\theta'\,d\theta'\right),\right. \\
&\qquad\qquad \left. z\left[\int_0^R \frac{r'}{(r'^2+z^2)^{3/2}}\,dr'\right]\left(\int_0^{2\pi}d\theta'\right)\right\rangle \\
&= \frac{\sigma}{4\pi\epsilon_0}\left\langle -\left[\int_0^R \frac{r'^2}{(r'^2+z^2)^{3/2}}\,dr'\right](0), -\left[\int_0^R \frac{r'^2}{(r'^2+z^2)^{3/2}}\,dr'\right](0), z\left[\int_0^R \frac{r'}{(r'^2+z^2)^{3/2}}\,dr'\right](2\pi)\right\rangle \\
&= \frac{\sigma}{4\pi\epsilon_0}\left\langle 0, 0, 2\pi z\int_0^R \frac{r'}{(r'^2+z^2)^{3/2}}\,dr'\right\rangle \\
&= \frac{\sigma z}{2\epsilon_0}\langle 0,0,1\rangle \int_0^R \frac{r'}{(r'^2+z^2)^{3/2}}\,dr' \\
&= \frac{\sigma z\hat{\mathbf{z}}}{2\epsilon_0}\int_0^R \frac{r'}{(r'^2+z^2)^{3/2}}\,dr'
\end{aligned}$$

Make the following substitution.

$$u = r'^2 + z^2$$

$$du = 2r'\,dr' \quad\rightarrow\quad \frac{du}{2} = r'\,dr'$$

The integral then becomes

$$\begin{aligned}
\mathbf{E} &= \frac{\sigma z\hat{\mathbf{z}}}{2\epsilon_0}\int_{z^2}^{R^2+z^2}\frac{1}{u^{3/2}}\left(\frac{du}{2}\right) \\
&= \frac{\sigma z\hat{\mathbf{z}}}{4\epsilon_0}\int_{z^2}^{R^2+z^2} u^{-3/2}\,du \\
&= \frac{\sigma z\hat{\mathbf{z}}}{4\epsilon_0}(-2u^{-1/2})\Big|_{z^2}^{R^2+z^2} \\
&= \frac{\sigma z\hat{\mathbf{z}}}{2\epsilon_0}(u^{-1/2})\Big|_{R^2+z^2}^{z^2}.
\end{aligned}$$

Therefore, the electric field at $\mathbf{r} = \langle 0,0,z\rangle$ is

$$\boxed{\mathbf{E} = \frac{\sigma z\hat{\mathbf{z}}}{2\epsilon_0}\left(\frac{1}{z} - \frac{1}{\sqrt{R^2+z^2}}\right).}$$

Observe that

$$\lim_{R\to\infty}\mathbf{E} = \lim_{R\to\infty}\frac{\sigma z\hat{\mathbf{z}}}{2\epsilon_0}\left(\frac{1}{z}-\frac{1}{\sqrt{R^2+z^2}}\right) = \frac{\sigma z\hat{\mathbf{z}}}{2\epsilon_0}\left(\frac{1}{z}-0\right) = \frac{\sigma}{2\epsilon_0}\hat{\mathbf{z}}.$$

In order to see what happens if $z \gg R$, rewrite the formula so that each term is a ratio of R and z, z being in the denominator, and use the binomial theorem.

$$\begin{aligned}
\mathbf{E} &= \frac{\sigma z\hat{\mathbf{z}}}{2\epsilon_0}\left(\frac{1}{z}-\frac{1}{z\sqrt{\frac{R^2}{z^2}+1}}\right) \\
&= \frac{\sigma\hat{\mathbf{z}}}{2\epsilon_0}\left(1-\frac{1}{\sqrt{\frac{R^2}{z^2}+1}}\right) \\
&= \frac{\sigma\hat{\mathbf{z}}}{2\epsilon_0}\left[1-\left(1+\frac{R^2}{z^2}\right)^{-1/2}\right] \\
&= \frac{\sigma\hat{\mathbf{z}}}{2\epsilon_0}\left[1-\sum_{k=0}^{\infty}\frac{\Gamma\left(-\frac{1}{2}+1\right)}{\Gamma(k+1)\Gamma\left(-\frac{1}{2}-k+1\right)}\left(\frac{R^2}{z^2}\right)^k\right] \\
&= \frac{\sigma\hat{\mathbf{z}}}{2\epsilon_0}\left[1-\sum_{k=0}^{\infty}\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(k+1)\Gamma\left(\frac{1}{2}-k\right)}\left(\frac{R}{z}\right)^{2k}\right] \\
&= \frac{\sigma\hat{\mathbf{z}}}{2\epsilon_0}\left[1-\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(1)\Gamma\left(\frac{1}{2}\right)}\left(\frac{R}{z}\right)^0-\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)\Gamma\left(-\frac{1}{2}\right)}\left(\frac{R}{z}\right)^2-\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(3)\Gamma\left(-\frac{3}{2}\right)}\left(\frac{R}{z}\right)^4-\cdots\right] \\
&= \frac{\sigma\hat{\mathbf{z}}}{2\epsilon_0}\left(1-1+\frac{R^2}{2z^2}-\frac{3R^4}{8z^4}+\cdots\right) \\
&= \frac{\sigma\hat{\mathbf{z}}}{2\epsilon_0}\left(\frac{R^2}{2z^2}-\frac{3R^4}{8z^4}+\cdots\right) \\
&= \left(\frac{Q}{\pi R^2}\right)\frac{\hat{\mathbf{z}}}{2\epsilon_0}\left(\frac{R^2}{2z^2}-\frac{3R^4}{8z^4}+\cdots\right)
\end{aligned}$$

If $z \gg R$, then all higher-order terms are negligible compared to R^2/z^2.

$$\mathbf{E} \approx \left(\frac{Q}{\pi R^2}\right)\frac{\hat{\mathbf{z}}}{2\epsilon_0}\left(\frac{R^2}{2z^2}\right) = \frac{1}{4\pi\epsilon_0}\frac{Q}{z^2}\hat{\mathbf{z}}$$

The lesson is that far away from the circular disk the electric field is the same as if it were a point charge.